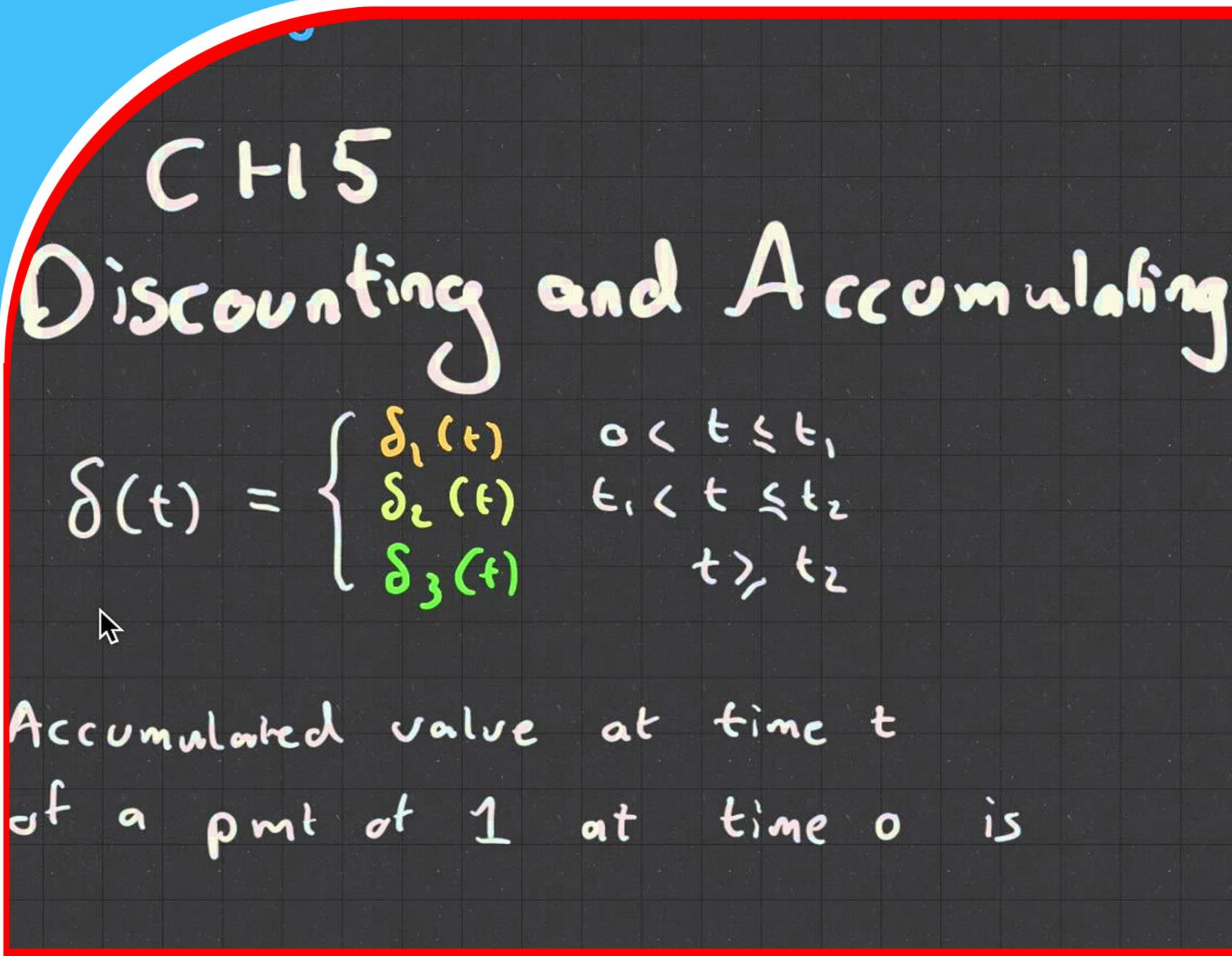
American Journal of
Statistics and Actuarial Science
(AJSAS)



Pay-As-You-Drive Insurance Pricing Model

Safoora Zarei
Alireza Fallahi

AJPO

American Journal of Statistics and Actuarial Science
ISSN 2520-4649 (online)
Vol.2, Issue 1, pp 1 - 9, 2020
www.ajpojournals.org

Pay-As-You-Drive Insurance Pricing Model

[1*]Safoora Zarei
[1]Lecturor, Shiraz Islamic Azad University, Shiraz, Iran
[*]Corresponding Author's Email: safoora.zarei19@gmail.com

[2]Alireza Fallahi
[2]Gradute Student, Department of Mathematics and Computer Science, Amirkabir University of Technology, Iran

Abstract

Purpose: Every time drivers take to the road, and with each mile that they drive, exposes themselves and others to the risk of an accident. Insurance premiums are only weakly linked to mileage, however, and have lump-sum characteristics largely. The result is too much driving, and too many accidents. The purpose of carrying out this research was to determine a model for calculating the premiums for Pay-As-You-Drive in Automobile insurances.

Methodology: To price Pay-As-You-Drvie auto insurance, we define a discounted collective risk model while the total number of claim has non-homogeneous Poisson distribution. By applying non-homogeneous Poisson distribution we can enter the mileage to the discounted collective risk model to the premiums for Pay-As-You-Drive in Automobile insurances. We apply the double Double Stochastic Poisson Process for modeling the the DCRM. The Double Stochastic Poisson Process provides flexibility by letting the intensity not only depend on time but also by allowing it to be a stochastic process.

Findings: By applying the doubly stochastic Poisson process to viewe the driver's milege in the model, we find the distribution of discounted collective risk model and present the expected value of total loss for calculating the premiums.

Policy recommendation: The current auto insurance pricing systems are inequitable because low-mileage drivers subsidize insurance costs for high-mileage drivers, and low-income people drive fewer miles on average. The study recommends a more efficient pricing systems model to find a good model for calculating the fair auto insurance premium.

Keywords: *Cox Process, Martingales, Aggregate Risk Models, PAYD, Actuarial Mathematics*.

American Journal of Statistics and Actuarial Science
ISSN 2520-4649 (online)
Vol.2, Issue 1, pp 1 - 9, 2020
www.ajpojournals.org

1. Introduction

In most developed countries, automobile insurance represents a considerable share of the yearly non-life premium collection. Because the majority of the incurred losses are usually very high. Therefore using this Insurance for car owners in some countries (such as Iran) is in force. Also, many attempts have been made in the actuarial literature to find a good model for calculating the premiums; for a review of the existing literature, we refer the interested readers, e.g., to Lemaire (1985), Lemaire (1995), or Frangos and Vrontos (2001).

In a general case there are two different approaches. The first approche is that the premium will be fixed for all of policyholders, and the second is the premium will be different for all of policyholders.

The first approach is unfair because, in this way, a policyholder who had an accident with a small size of loss (or did not have an accident) is being unfairly disadvantaged to a policyholder who had an accident with a big size of the loss. The second method is the right way and is the base of Bonus Malus Systems (BMS). BMS penalizes insurers responsible for one or more accidents by premium surcharges (or maluses), and rewarding claim-free policyholders by awarding them discounts (or bonuses). This is a very efficient way of classifying policyholders into cells according to their risk (see Denuit and Dhaene, 2001). A BMS calculates the premium applicable as a base premium, adjusted by a quantity (the bonus or malus) which depends on previous claims experience (see Taylor, 1997). The three systems for adjusting the premiums are: BMS based on the frequency component, BMS based on the frequency and severity component, and BMS based on the frequency component, severity component and individual characteristics (such as sex, age, type of car, location and ...).

It is obvious that the third system is a generalization of two other systems (see Frangos and Vrontos, 2001). Moreover, Mahmoudvand and Hassani (2009) have introduced a new generalization of BMS that covers all three types of BMS that mentioned above.

Recently, some researchers believe that the current lump-sum pricing of auto insurance is inefficient and inequitable (see Bordoff and Noel, 2008). Even according to the optimal BMS, drivers who are similar in claims and individual characteristics pay nearly the same premiums if they drive five thousand or fifty thousand miles a year. Considerable research indicates that annual crash rates and claim costs tend to increase with annual vehicle mileage (see Litman, 2009). This pricing system is inequitable because low-mileage drivers subsidize insurance costs for high-mileage drivers, and low-income people drive fewer miles on average.

A better approach is simple and obvious: pay-as-you-drive (PAYD) auto insurance. With PAYD, insurance premiums would be priced per mile driven. All other risk factors will still be taken into account so that a high-risk driver would pay a higher per-mile premium than a low-risk driver. With insurance costs that vary with miles driven, people would be able to save money by reducing their driving, and this incentive would lead to fewer driving-related harms. PAYD would also be more equitable because it would eliminate the cross-subsidization of

American Journal of Statistics and Actuarial Science
ISSN 2520-4649 (online)
Vol.2, Issue 1, pp 1 - 9, 2020
www.ajpojournals.org

insurance costs from low-mileage to high-mileage drivers.

Parry (2005) shows that PAYD is slightly more efficient than a simple tax on vehicle miles traveled (VMT) for a given fuel reduction and even performs reasonably well relative to a fully optimized VMT tax. Although PAYD insurance has long been advocated by transportation planners, little attention has been given to the precise design of a distance-based pricing system.

Remain of this paper is as follows: In Section 2, we present the discounted collective risk models and some useful Theorems on it. In Section 3, we enter the mileage to the discounted collective risk model to consider PAYD. Finally, some conclusions are presented in Section 4.

2. Discounted Collective Risk Models

Although the collective risk model seems to have many advantages, one of its drawbacks is that it overlooks the arrival time of claims and the effect of the interest rate. In property liability insurance contracts, there is always a time lag between the premium payment and claims arrival time. During this time lags, the insurer earns investment income on the unexpanded component of the premium, which is not involved in the collective risk model. So the insured is eligible to have some of this investment profit during the policy coverage period.

Definition 1 *A discounted collective risk model (DCRM) in a specific period of time* $(0;t]$, *represents the total loss,* Z_t, *as the sum of a random number claims,* $N(t)$, *of individual present value payment amounts* $(X_1, X_2, \dots, X_{N(t)})$ *respect to arrival times* $(W_1, W_2, \dots, W_{N(t)})$ *and constant force of interest* δ *as follows:*

$$Z_t = \sum_{i=1}^{N(t)} X_i \, e^{-\delta W_i} \tag{1}$$

Where:

- Individual claims X_i are independent and identically distributed,
- $N(t)$ and X_i are independent, and
- $Z_t = 0$ when $N(t) = 0$.

The DCRM defined as above has several interesting and useful properties. At first, it incorporates investment income into the pricing model. Moreover, it provides a better model for property and liability insurance in which the interval between premium payments and claim payments is a significant factor. Therefore, insurers can present long term insurance products in the property and liability insurance market. One important quantity is the expected value of Z_t, which can be interpreted as the net premium amount needed to cover insurance liability on its becoming due without paying any expenses or contingent charges. We calculate this expect value in an important special case of DCRM where $N(t)$ has non-homogeneous Poisson distribution.

Using the martingale approach, many interesting results can be obtained; refer to Gerber and Shiu (1997) for a thorough discussion. In the following theorem, we use a similar technique to find the moment generating a function of Z_t.

Theorem 1 *Let* $M_{Z_t}(u)$ *denote the moment generating function (m.g.f) of* Z_t *defined by relation 12 and let* $N(t) \sim Possion(\lambda(t))$, *then*

$$M_{Z_t}(u) = exp\left\{-\int_0^t \lambda(s)\left(1 - M_X\left(u\, e^{-\delta s}\right)\right)\right\}, \tag{2}$$

where $M_X(.)$ is m.g.f of X.

Proof. Consider process $\{M_t\}_{t>0} = \left\{\frac{e^{u Z_t}}{g(t,u)}\right\}_{t>0}$ where $g(t,u)$ is a function to be determined later and satisfies in the initial condition $g(0,u) = 1$. We first seek a value of $g(t,u)$ such that $\{M_t\}_{t>0}$ is a martingale. To do this, we note that based on the properties of martingale, M_t must satisfy in the following relation (for all $h > 0$):

$$E\left[\frac{M_{t+h}}{M_t} | M_t = m_t\right] = 1.$$

In this case, we have:

$$E\left[e^{u \sum_{i=N(t)+1}^{N(t+h)} X_i e^{-\delta W_i}}\right] = \frac{g(t+h,u)}{g(t,u)}$$

Now by the rule of Iterated expectation, it can be shown that

$$\begin{aligned} &E\left[E\left[e^{u\sum_{i=N(t)+1}^{N(t+h)} X_i e^{-\delta W_i}} | N(t+h) - N(t) = k\right]\right] \\ &= \sum_{k=0}^{\infty} E\left[e^{u\sum_{i=N(t)+1}^{N(t+h)} X_i e^{-\delta W_i}} | N(t+h) - N(t) = k\right] Pr[N(t+h) - N(t) = k] \\ &= \frac{g(t+h,u)}{g(t,u)} \end{aligned} \tag{3}$$

Based on the properties of the Poisson process we can rewrite 3 as follows,

$$\begin{aligned} &\sum_{k=0}^{\infty} E\left[e^{u\sum_{i=N(t)+1}^{N(t+h)} X_i e^{-\delta W_i}} | N(t+h) - N(t) = k\right] Pr[N(t+h) - N(t) = k] \\ &= (1 - \lambda(t)h) + E\left[e^{uX_{N(t+h)} e^{-\delta(t+h)}} | N(t+h) - N(t) = k\right] \lambda(t)h + o(h) \\ &= (1 - \lambda(t)h) + M_X\left(ue^{-\delta(t+h)}\right)\lambda(t)h + o(h) = \frac{g(t+h,u)}{g(t,u)} \end{aligned}$$

Where $o(h)$ is a generic function that goes to zero faster than h when h goes to zero. By a few simplifications we have:

$$-\lambda(t)\left(1-M_X\left(ue^{-\delta(t+h)}\right)\right)+\frac{o(h)}{h}=\frac{g(t+h,u)-g(t,u)}{h.g(t,u)} \quad (4)$$

Taking limits as $h \to 0$ in the above relation, we have:

$$\frac{d}{dt}lng(t,u)=\left(1-M_X\left(ue^{-\delta t}\right)\right) \quad (5)$$

Now it is sufficient to show that $M_{Z_t}=g(t,u)=g(t,u)$. It follows from the initial condition $g(0,u)=1$ that $M_0=1$. Moreover based on the properties of martingale, we have $E(M_t)=E(M_0)=1$, which completes the proof.

Corollary 1 *Suppose in the* $DCRM(1)$, $N(t) \sim Poisson(\lambda t)$, *then*

$$E[Z_t]=\frac{\mu_1\lambda}{\delta}\left(1-e^{-\delta t}\right), \quad (6)$$

where $\mu_1=E(X)$.

Corollary 2 *Process* $\{A_t\}_{t>0}=\left\{Z_t-\frac{\mu_1\lambda}{\delta}\left(1-e^{-\delta t}\right)\right\}_{t>0}$ *is a martingale.*

Let us now consider the discrepancy between the obtained premiums based on the collective risk model, and by the DCRM equation (6). Recall that if S_t shows the collective risk model, then $S_t=\sum_{i=1}^{N(t)} X_i$ In fact S_t is a special case of relation (6) when the $\delta \to 0$. It is easy to see that,

$$\lim_{\delta\to 0} E[Z_t]=E[S_t].$$

Another special case is when $t \to \infty$. In this case:

$$\lim_{t\to\infty} E[Z_t]=\lim_{t\to\infty}\frac{\mu_1\lambda}{\delta}\left(1-e^{-\delta t}\right), \quad (7)$$

which can be interpreted as a single net premium for a perpetuity that continuously pays $\mu_1\lambda$.

Moreover, note that if $\delta \to \infty$, then $E[Z_t] \to 0$, and when $t \to 0$, then $E[Z_t] \to 0$, which are reasonable results.

Corollary 3 *Consider the DCRM described in Dis_Coll, if* $N(t) \sim Poisson(\lambda t)$, *then*

$$Var[Z_t]=\frac{\mu_2\lambda}{2\delta}\left(1-e^{-2\delta t}\right) \quad (8)$$

where $\mu_2=E(X^2)$.

Corollary 4 *The process* $\{B_t\}_{t>0} = \left\{\left(Z_t - \frac{\mu_1\lambda}{\delta}\left(1-e^{-\delta t}\right)\right)^2 - \frac{\mu_2\lambda}{2\delta}\left(1-e^{-2\delta t}\right)\right\}_{t>0}$ *is a martingale.*

Example 1 *In the discounted collective risk, let claim sizes are exponentially distributed with mean then the m.g.f is given by:*

$$M_{Z_t}(u) = E[e^{uZ_t}] = \left(\frac{1-\beta u e^{-\delta t}}{1-\beta u}\right)^{\frac{\lambda}{\delta}}. \tag{9}$$

It follows from this example that:

$$\lim_{\delta\to 0} M_{Z_t}(u) = exp\left(\frac{\lambda t u\beta}{1-u\beta}\right) \tag{10}$$

Moreover by limiting when t tend to in nity we have:

$$\lim_{\delta\to 0} M_{Z_t}(u) = (1-u\beta)^{-\frac{\lambda}{\delta}} \tag{11}$$

which are coincide to the results that Gerber (1979) has obtained.

3. Modeling PAYD by means of DCRM

In PAYD Insurance, we have lots of information with the help of the GPS system, and our goal is to set a premium based on the distance that a person travels during a year. By presenting this new product, insurers are facing a new source of risk, which is a random premium. We do not have an exact amount of kilometers that the driver will drive. Let $d(t)$ is mileage to time t. We would like to define a DCRM that consider $d(t)$. To do this, there is two different approaches to consider $d(t)$ in the models.

The first approach is using double subordinated model defined by Sato (1999); Shirvani et al. (2019). Subordination is an often used stochastic process in modeling asset prices. Applications of subordination model and Lévy processes arise in science and engineering, e.g., quantum mechanics, insurance, economics, finance, biomathematics, etc.[1] Shirvani et al. (2019) introduced the theory of multiple subordinated model to modeling the tail behavior of stock market returns.[2]

To apply the double subordinator models for modeling the the DCRM, Let X_i and $d(t) = U(t)$, be Lévy subordinators.[3] Then, the double subordinator $V(t) = X_i(U(t))$ represent the individual claims when the subordinator $d(t) = U(t)$ is the miles mileage to time t. Therefore, the DCRM model is:

---

[1] See Michna (2010), Sims et al. (2012), Lefévre and Picard (2013), Morales (2007), Levajkovic et al. (2016), Shirvani and Volchenkov (2019), and Shirvani et al. (2019).

[2] See also Shirvani et al. (2019).

[3] A Lévy subordinator is a Lévy process with an increasing sample path (see Sato, 1999).

American Journal of Statistics and Actuarial Science
ISSN 2520-4649 (online)
Vol.2, Issue 1, pp 1 - 9, 2020 www.ajpojournals.org

$$Z_t = \sum_{i=1}^{N(t)} X_i(U(t))\ e^{-\delta W_i}. \tag{12}$$

However, this model for DRCM is a new method, which is beyond the scope of this paper.

The second approach for modeling $d(t)$ is to use the Double Stochastic Poisson Process (DSPP). Since the goal of this paper is using DSPP processes, let us give a brief definition of DSPP. We notice that many alternative definitions of a DSPP can be given (see Grandell,

1976; Bremaud, 1981).

Definition 2 *A DSPP* $\{N(t): t > t_0\}$ *with intensity stochastic process* $\{\lambda(t, d(t)): t > t_0\}$ *is defined as a conditioned Poisson process which intensity is the process*$\{\lambda(t, d(t))): t > t_0\}$ *given the information process*$\{d(t): t > t_0\}$.

The DSPP, or Cox process, provides flexibility by letting the intensity not only depend on time but also by allowing it to be a stochastic process. Therefore, the doubly stochastic Poisson process can be viewed as a two-step randomization procedure. An intensity process $\{\lambda(t, d(t)): t > t_0\}$ is used to generate another process $\{N(t): t > t_0\}$ by acting as its intensity. If $\{\lambda(t, d(t)): t > t_0\}$ is deterministic, then $\{N(t): t > t_0\}$ is a nonhomogeneous Poisson process. If $\{\lambda(t, d(t)): t > t_0\} = \lambda$ for some positive random variable λ, then $\{N(t): t > t_0\}$ is a mixed Poisson process.

Theorem 2 *Let* $M_{Z_t}(u)$ *denote the m.g.f of* Z_t *defined by relation (12) and let* $N(t)$ *is a DSPP with intensity process* $(\lambda(t, d(t)))$, *then*

$$M_{Z_t}(u) = E\left[exp\left\{-\int_0^t \lambda(s, d(s))\left(1 - M_x\left(ue^{-\delta s}\right)\right) dt\right\}\right] \tag{13}$$

where $M_X(.)$ is m.g.f of X.

Proof. Conditioning on $(\lambda(t, d(t)))$ and using Theorem (1) results can be obtained.

Corollary 5 *Under conditions of the Theorem (2) we have*

$$E[Z_t] = \mu_1 E\left[\int_0^t \lambda(s, d(s)) e^{-\delta s} ds\right]. \tag{14}$$

4. Conclusion

With PAYD, insurance premiums would be priced per mile driven. All other risk factors will still be taken into account so that a high-risk driver would pay a higher per-mile premium than a low-risk driver. With insurance costs that vary with miles driven, people would be able to save money by reducing their driving, and this incentive would lead to fewer driving-related harms. PAYD would also be more equitable because it would eliminate the cross-subsidization of insurance costs from low-mileage to high-mileage drivers.

As we said, the DSPP provides flexibility by letting the intensity not only depend on time but also by allowing it to be a stochastic process. Therefore, the doubly stochastic Poisson process can be viewed as a two-step randomization procedure. We show that it is possible to model PAYD by using DSPP.

References

Bordoff, J. and P. Noel (2008). *Pay-As-You-Drive Auto Insurance: A Simple Way to Reduce Driving-Related Harms and Increase Equity*. The Brooking Institution.

Bremaud, P. (1981). *Point Processes and Queues: Martingale Dynamics*. New York: Springer Verlag.

Denuit, M. and J. Dhaene (2001). Bonus-malus scales using exponential loss functions. *Blätterder DGVFM 25*, 13–27.

Frangos, N. and S. Vrontos (2001). Design of optimal bonus malus systems with a frequency and severity component on an individual base in automobile insurance. *Astin Bulletin 31*(1), 1–22.

Gerber, H. and S. Shiu (1997). The joint distribution of the time of ruin, the surplus immediately before ruin, and the deficit at ruin. *Insurance: Mathematics and Economics 21*(2), 129–137.

Grandell, P. (1976). *Doubly Stochastic Poisson Processes*. Berlin: Springer Verlag.

Lefévre, C. and P. Picard (2013). Ruin time and severity for a lévy subordinator claim process: A simple approach. *Journal of Risks 1*, 192–212.

Lemaire, J. (1985). *Automobile Insurance: Actuarial Models*. Netherlands: Kluwer Nijholff.

Lemaire, J. (1995). *Bonus Malus Systems in Automobile Insurance*. Boston: Kluwer Academic Publisher.

Levajkovic, T., H. Mena, and M. Zarfl (2016). Lévy processes, subordinators and crime modeling. *Novi Sad J. Maths 46*, 65–86.

Litman, T. (2009). Pay-as-you-drive pricing for insurance affordability. *Victoria Transport Policy Institute*, 1–19.

Mahmoudvand, R. and H. Hassani (2009). Generalized bonus-malus systems with an infrequence and severity component on an individual basis in automobile insurance. *ASTIN Bulletin 39*, 307–315.

Michna, Z. (2010). Ruin probability on a finite time horizon. *Math. Econ. 6*, 65–74.

Morales, M. (2007). On the expected discounted penalty function for a perturbed risk process driven by a subordinator. *Insurance. Math. Econ. 40*, 293–301.

Parry, T. (2005). *Is Pay-As-You-Drive Insurance a Better Way to Reduce Gasoline than*

Gasoline Taxes? Resources for the Future.

Sato, K.-I. (1999). *Lévy Processes and Infinitely Divisible Distributions*. Cambridge: Cambridge studies in advanced mathematics.

Shirvani, A., Y. Hu, S. Rachev, and F. Fabozzi (2019). Mixed levy subordinated market model and implied probability weighting function. *arXiv:1910.05902*.

Shirvani, A., S. Rachev, and F. Fabozzi (2019). Multiple subordinated modeling of asset returns. *arXiv:1907.12600*.

Shirvani, A., S. Stoyanov, F. Fabozzi, and S. Rachev (2019). Equity premium puzzle or faulty economic modelling? *arXiv:1909.13019*.

Shirvani, A. and D. Volchenkov (2019). A regulated market under sanctions: On tail dependence between oil, gold, and Tehran stock exchange index. *Journal of Vibration Testing and System Dynamics 3*(3), 297–311.

Sims, D., N. Humphries, R. Bradford, and B. Bruce (2012). Lévy flight and Brownian search patterns of a free-ranging predator reflect different prey field. *J Anim Ecol 81*(2), 432–442.

Taylor, G. (1997). Setting a bonus-malus scale in the presence of other rating factors. *ASTIN Bulletin 27*, 319–327.