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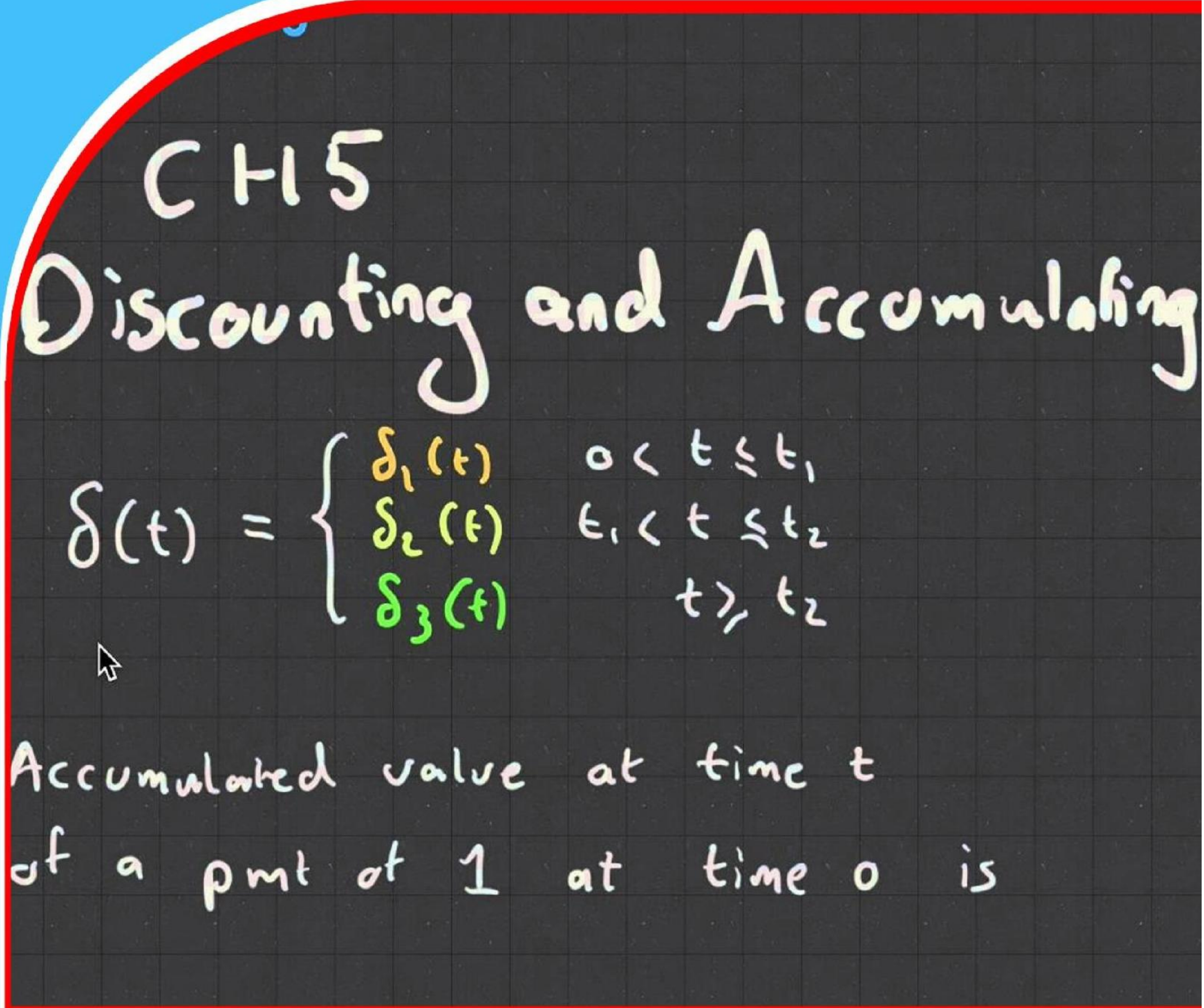

THE RELATIONSHIP AMONG THE THREE DISTRIBUTIONS: BINOMIAL, POISSON AND NORMAL DISTRIBUTION

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Abstract

The relationship among the three distributions: Binomial, Poisson and Normal Distribution are of significant interest in many application contexts. Most of the papers in the literature have focused on the relationship among the three distributions based on the definition. In contrast, this paper proposes a simpler approach on how to show the relationship between the Normal distribution and Poisson distribution based on the moment generating function. The study has also proposed the method on how to show that the moment generating function for Poisson distribution can be driven from the moment generating function for Normal distribution when $t = 0$, and the moment generating function for Normal distribution is the same as the Probability Mass Function (PMF) for Poisson distribution. The study has proposed on how to show that the moment generating function for Poisson distribution can be driven from the moment generating function for Binomial distribution.

Keywords: *Binomial Distribution, Poisson Distribution, Normal Distribution, Moment Generating Function*

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1.0 INTRODUCTION

The study will highlight the relationship among Binomial Distribution, Poisson and Normal Distribution based on the definition, moment generating function and application.

2. THE RELATIONSHIP AMONG THE THREE DISTRIBUTIONS BASED ON THE DEFINITION AND APPLICATION

By definition and properties of Binomial and Poisson distribution, the two distributions are related. Poisson distribution is best used when the sample size is large (> 10) while the Binomial distribution is useful when the sample size is small (< 10). Moreover, the mean for Binomial distribution is used to find the mean for Poisson distribution. That is λ (mean of the Poisson distribution) $= np$ (mean of the Binomial distribution). Despite the Normal Distribution being a continuous distribution, it is also related to the Binomial and Poisson distribution. The relationship comes in when the continuity correction factor is applied. A continuity correction factor is used when you use a continuous function to approximate a discrete one. In this case when the Normal distribution is used to approximate Binomial and Poisson distribution then the continuity correction factor is applied. In order to use same set of all possible values of μ and σ^2, the value of x should be standardised so that the mean $\mu = 0$ and $\sigma^2 = 1$. The standardised normal variable value of x is called Z – score and $Z = \frac{X-\mu}{\sigma}$. For Binomial distribution $\mu = np$ and $\sigma^2 = npq$. For Poisson distribution, the mean and variance are the same. That is $\mu = np = \lambda$ and $\sigma^2 = npq = \lambda$

From the above equations, if n is large enough the random variable Z is approximately normally distributed and we can use the normal distribution to approximate the Poisson or Binomial distribution. For discrete distributions (Binomial and Poisson distribution), the probability of finding more than $(>)$ is the same as one minus the probability of finding less than or equal to $(\leq)$. The probability of finding at least $(\geq)$ is the same as one minus the probability of finding less than $(<)$ (using the complementary rule of probability might be easy). That is $P(X > r) = 1 - P(X \leq r)$ and $P(X \geq r) = 1 - P(X < r)$ for $r = 0,1,2, \dots, n$

3. MOMENT GENERATING FUNCTIONS FOR THE THREE DISTRIBUTIONS

According to (Hwei , 1997), the moment generating function of a random variable X is defined by $M_X(t) = E(e^{tx}) = \sum_{x=0}^{n} e^{tx} P(X = x)$ for discrete distributions and $M_X(t) = E(e^{tx}) = \int_{-\infty}^{\infty} e^{tx} f(x)dx$ for continuous distributions where t is the real variable.

3.1 The Moment Generating Function for the Poisson distribution

$$M_X(t) = E(e^{tx}) = \sum_{x=0}^{n} e^{tx} P(X = x)$$

Where $P(X = x) = \frac{\lambda^x e^{-\lambda}}{x!}$

$$M_X(t) = \sum_{x=0}^{n} e^{tx} \frac{\lambda^x e^{-\lambda}}{x!}$$
$$M_X(t) = e^{-\lambda} \sum_{x=0}^{n} e^{tx} \frac{\lambda^x}{x!}$$
$$M_X(t) = e^{-\lambda} \sum_{x=0}^{n} \frac{(\lambda e^t)^x}{x!}$$
$$and$$
$$\sum_{x=0}^{n} \frac{(\lambda e^t)^x}{x!} = 1 + \frac{\lambda e^t}{1!} + \frac{\lambda^2 e^{2t}}{2!} + \frac{\lambda^3 e^{3t}}{3!} + \frac{\lambda^4 e^{4t}}{4!} + \cdots$$
$$and$$
$$1 + \frac{\lambda e^t}{1!} + \frac{\lambda^2 e^{2t}}{2!} + \frac{\lambda^3 e^{3t}}{3!} + \frac{\lambda^4 e^{4t}}{4!} + \cdots$$

Is the Maclaurin series expansion of the form $e^{\lambda e^t}$. $_t$Therefore,
$$M_X(t) = e^{-\lambda}.e^{\lambda e}$$
$$M_X(t) = e^{\lambda(e^t - 1)}$$

3.2 The Moment Generating Function for the Binomial Distribution

$$M_X(t) = E(e^{tx}) = \sum_{x=0}^{n} e^{tx} P(X = x)$$

Where $P(X = x) = nC_x p^x q^{n-x}$
$$M_X(t) = \sum_{x=0}^{n} e^{tx}\, nC_x p^x q^{n-x}$$
$$M_X(t) = \sum_{x=0}^{n} \frac{e^{tx} p^x q^{n-x}}{(n-x)!\,x!}$$
$$M_X(t) = q^n + \frac{n!}{(n-1)!\,1!} e^t p q^{n-1} + \frac{n!}{(n-2)!\,2!} e^{2t} p^2 q^{n-2} + \cdots$$
$$M_X(t) = q^n + n e^t p q^{n-1} + \frac{n(n-1)}{2!} e^{2t} p^2 q^{n-2} + \frac{n(n-1)(n-2)}{3!} e^{3t} p^3 q^{n-3} + \cdots$$
$$and$$
$q^n + n e^t p q^{n-1} + \frac{n(n-1)}{2!} e^{2t} p^2 q^{n-2} + \frac{n(n-1)(n-2)}{3!} e^{3t} p^3 q^{n-3} + \cdots$ is the binomial expansion of the form

$$q^n \left[1 + \frac{pe^t}{q}\right]^n = [q + pe^t]^n$$

$$M_X(t) = [q + pe^t]^n$$

3.3 The Moment Generating Function for the Normal Distribution

$$M_X(t) = E(e^{tx}) = \int_{-\infty}^{\infty} e^{tx} f_x(x)dx$$

$$M_X(t) = \int_{-\infty}^{\infty} e^{tx} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2} dx$$

$$M_X(t) = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{tx}. e^{-\frac{1}{2\sigma^2}(x^2-2x\mu+\mu^2)} dx$$

$$M_X(t) = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}(x^2-2x\mu+\mu^2)+tx} dx$$

$$M_X(t) = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{-\frac{\mu^2}{2\sigma^2}}. e^{-\frac{1}{2\sigma^2}(x^2-2x(\mu+\sigma^2 t)} dx$$

$$M_X(t) = \frac{e^{-\frac{\mu^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}(x^2-2x(\mu+\sigma^2 t)} dx$$

Using completing of square method

$$x^2 - 2x(\mu + \sigma^2 t = \left(x - (\mu + \sigma^2 t)\right)^2 - (\mu^2 + 2\mu\sigma^2 t + \sigma^4 t^2)$$

$$M_X(t) = \frac{e^{-\frac{\mu^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}\left(\left(x-(\mu+\sigma^2 t)\right)^2-(\mu^2+2\mu\sigma^2 t+\sigma^4 t^2)\right)} dx$$

$$M_X(t) = \frac{e^{-\frac{\mu^2}{2\sigma^2}}. e^{-\frac{1}{2\sigma^2}\left(-(\mu^2+2\mu\sigma^2 t+\sigma^4 t^2)\right)}}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}\left(\left(x-(\mu+\sigma^2 t)\right)^2\right)} dx$$

$$M_X(t) = \frac{e^{\frac{1}{2\sigma^2}(2\mu\sigma^2 t+\sigma^4 t^2)}}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}\left(\left(x-(\mu+\sigma^2 t)\right)^2\right)} dx$$

$$M_X(t) = \frac{e^{\frac{2\sigma^2}{2\sigma^2}\left(\mu t+\frac{1}{2}\sigma^2 t^2\right)}}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}\left(\left(x-(\mu+\sigma^2 t)\right)^2\right)} dx$$

$$M_X(t) = \frac{e^{\mu t+\frac{1}{2}\sigma^2 t^2}}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}\left(\left(x-(\mu+\sigma^2 t)\right)^2\right)} dx$$

Let $z = x - (\mu + \sigma^2 t)$
$dz = dx$

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$$M_X(t) = \frac{e^{\mu t + \frac{1}{2}\sigma^2 t^2}}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{-\frac{Z^2}{2\sigma^2}} dz$$

Using gamma function approach to evaluate

$$\int_{-\infty}^{\infty} e^{-\frac{Z^2}{2\sigma^2}} dz$$

$$\int_{-\infty}^{\infty} e^{-\frac{Z^2}{2\sigma^2}} dx = 2\int_{0}^{\infty} e^{-\frac{Z^2}{2\sigma^2}} dz$$

Let $t = \frac{Z^2}{2\sigma^2}$

$$z^2 = 2\sigma^2 t$$

$$z = \sqrt{2\sigma^2}\, t^{\frac{1}{2}}$$

$$dz = \frac{\sqrt{2\sigma^2}}{2} t^{-\frac{1}{2}} dt$$

$$2\int_{0}^{\infty} e^{-\frac{Z^2}{2\sigma^2}} dx = 2\int_{0}^{\infty} \frac{\sqrt{2\sigma^2}}{2} t^{-\frac{1}{2}} e^{-t} dt$$

$$2\int_{0}^{\infty} e^{-\frac{Z^2}{2\sigma^2}} dx = \sqrt{2\sigma^2}\int_{0}^{\infty} t^{-\frac{1}{2}} e^{-t} dt = \sqrt{2\sigma^2}\, \Gamma\left(\frac{1}{2}\right)$$

and $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

$$2\int_{0}^{\infty} e^{-\frac{Z^2}{2\sigma^2}} dx = \sqrt{2\sigma^2}\int_{0}^{\infty} t^{-\frac{1}{2}} e^{-t} dt = \sqrt{2\sigma^2}\, \Gamma\left(\frac{1}{2}\right) = \sqrt{2\sigma^2}.\sqrt{\pi} = \sqrt{2\pi\sigma^2}$$

$$M_X(t) = \frac{e^{\mu t + \frac{1}{2}\sigma^2 t^2}}{\sqrt{2\pi\sigma^2}}.\sqrt{2\pi\sigma^2}$$

$$M_X(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2}$$

4. THE RELATIONSHIP BETWEEN THE NORMAL DISTRIBUTION AND POISSON DISTRIBUTION BASED ON THE MOMENT GENERATING FUNCTION

Since the mean and variance for the Poisson distribution are the same, $\mu = \lambda$ and $\sigma^2 = \lambda$

$$M_X(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2}$$

$$M_X(t) = e^{\lambda t + \frac{1}{2}\lambda t^2}$$

$$M_X(t) = e^{\lambda\left(t + \frac{1}{2}t^2\right)}$$

Adding 1 and subtracting 1 to $t+\frac{1}{2}t^2$ we have $\left(1+t+\frac{1}{2}t^2\right)-1$

$$M_X(t)=e^{\lambda\left(\left(1+t+\frac{1}{2}t^2\right)-1\right)}$$

$1+t+\frac{1}{2}t^2$ is the Maclaurin series expansion of the form e^t

$$M_X(t)=e^{\lambda((e^t)-1)}$$

$$M_X(t)=e^{\lambda(e^t-1)}$$

Therefore, it has been shown that moment generating function for Poisson distribution can be driven from the moment generating function for Normal distribution.

5. THE RELATIONSHIP BETWEEN THE BINOMIAL AND POISSON DISTRIBUTION BASED ON THE MOMENT GENERATING FUNCTION

$$_{n\to\infty}\quad _X\ t\ = _{n\to\infty}\quad q+pe$$

$$=\lim_{n\to\infty}\left[q^n+ne^tpq^{n-1}+\frac{n(n-1)}{2!}e^{2t}p^2q^{n-2}+\frac{n(n-1)(n-2)}{3!}e^{3t}p^3q^{n-3}+\cdots\right]$$

Since $M_X(t)=[q+pe^t]^n$, then lim

M ( ) lim [^{t}]^n

Since $q=1-p$ then,

$$\lim_{n\to\infty}M_X(t)=\lim_{n\to\infty}\left[(1-p)^n+ne^tp(1-p)^{n-1}+\frac{n\ n-}{2!}e^{2t}p^2(1-p)^{n-2}+\frac{n(n-1)(n-2)}{3!}e^{3t}(1-p)^3+\cdots\right]$$

$$p=\frac{\lambda}{n},np=\lambda$$

$$\lim_{n\to\infty}M_X(t)=\lim_{n\to\infty}\left[(1-p)^n+npe^t\left(1-\frac{\lambda}{n}\right)^{n-1}+\frac{n^2p^2(1-\frac{1}{n})}{2!}e^{2t}\left(1-\frac{\lambda}{n}\right)^{n-2}+\frac{n^3p^3(1-\frac{1}{n})(1-\frac{2}{n})}{3!}e^{3t}\left(1-\frac{\lambda}{n}\right)^3+\cdots\right]$$

$$\lim_{n\to\infty}M_X(t)=\lim_{n\to\infty}\left[\left(1-\frac{\lambda}{n}\right)^n+\lambda e^t\left(1-\frac{\lambda}{n}\right)^{n-1}+\frac{\lambda^2(1-\frac{1}{n})}{2!}e^{2t}\left(1-\frac{\lambda}{n}\right)^{n-2}+\frac{\lambda^3(1-\frac{1}{n})(1-\frac{2}{n})}{3!}e^{3t}\left(1-\frac{\lambda}{n}\right)^3+\cdots\right] \qquad (\quad 1)$$

$$\lim_{n\to\infty} M_X(t) = \lim_{n\to\infty} \left[\left(1-\frac{\lambda}{n}\right)^n + \lambda e^t \frac{\left(1-\frac{\lambda}{n}\right)^n}{\left(1-\frac{\lambda}{n}\right)^1} + \frac{\lambda^2(1-\frac{1}{n})}{2!} e^{2t} \frac{\left(1-\frac{\lambda}{n}\right)^n}{\left(1-\frac{\lambda}{n}\right)^2} + \frac{\lambda^3(1-\frac{1}{n})(1-\frac{2}{n})}{3!} e^{3t} \frac{\left(1-\frac{\lambda}{n}\right)^n}{\left(1-\frac{\lambda}{n}\right)^3} + \cdots\right]$$

$$\lim_{n\to\infty} M_X(t) = \lim_{n\to\infty} \left(1-\frac{\lambda}{n}\right)^n + \lambda e^t \lim_{n\to\infty} \frac{\left(1-\frac{\lambda}{n}\right)^n}{\left(1-\frac{\lambda}{n}\right)^1} + \frac{\lambda^2 \lim_{n\to\infty}(1-\frac{1}{n})}{2!} e^{2t} \lim_{n\to\infty} \frac{\left(1-\frac{\lambda}{n}\right)^n}{\left(1-\frac{\lambda}{n}\right)^2} + \frac{\lambda^3 \lim_{n\to\infty}(1-\frac{1}{n})(1-\frac{2}{n})}{3!} e^{3t} \lim_{n\to\infty} \frac{\left(1-\frac{\lambda}{n}\right)^n}{\left(1-\frac{\lambda}{n}\right)^3} + \cdots$$

Since $\lim_{n\to\infty}\left(1+\frac{\lambda}{n}\right)^n = e^{\lambda}$ then

$$\lim_{n\to\infty} M_X(t) = e^{-\lambda} + \lambda e^t e^{-\lambda} + \frac{\lambda^2}{2!} e^{2t} e^{-\lambda} + \frac{\lambda^3}{3!} e^{3t} e^{-\lambda} + \frac{\lambda^4}{4!} e^{4t} e^{-\lambda} + \cdots$$

$$\lim_{n\to\infty} M_X(t) = e^{-\lambda}\left(1 + \lambda e^t + \frac{\lambda^2}{2!} e^{2t} + \frac{\lambda^3}{3!} e^{3t} + \frac{\lambda^4}{4!} e^{4t} + \cdots\right)$$

$1 + \lambda e^t + \frac{\lambda^2}{2!} e^{2t} + \frac{\lambda^3}{3!} e^{3t} + \frac{\lambda^4}{4!} e^{4t} + \cdots$ is the Maclaurin series expansion of the form $e^{\lambda et}$

$$\lim_{n\to\infty} M_X(t) = e^{-\lambda} e^{\lambda e^t}$$

$\lim_{n\to\infty} M_X(t) = e^{\lambda(et-1)}$. Which is the moment generating function for Poisson distribution.

Therefore, it has been shown that moment generating function for Poisson distribution can be driven from the moment generating function for Binomial distribution.

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6. THE RELATIONSHIP BETWEEN THE MOMENT GENERATING FUNCTION OF NORMAL DISTRIBUTION AND THE PROBABILITY MASS FUNCTION (PMF) OF POISSON DISTRIBUTION

$$[M_X(t)]_{t=0} = P(X = x) = \frac{\lambda^x e^{-\lambda}}{x!}$$

$$Since\ \ e^{\mu t + \frac{1}{2}\sigma^2 t^2} = e^{\lambda(e^t - 1)}$$

$$[M_X(t)]_{t=0} = \left[e^{\mu t + \frac{1}{2}\sigma^2 t^2}\right]_{t=0}$$

$$[M_X(t)]_{t=0} = \left[e^{\lambda(e^t - 1)}\right]_{t=0}$$

$$[M_X(t)]_{t=0} = \left[e^{\lambda e^t - \lambda}\right]_{t=0}$$

$$[M_X(t)]_{t=0} = e^{-\lambda}\left[e^{\lambda e^t}\right]_{t=0}$$

$$[Mx(t)]_{t=0} = e{-\lambda}(e\lambda)$$

Using the Maclaurin series expansion, $e^{\lambda} = 1 + \frac{\lambda}{1!} + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \frac{\lambda^4}{4!} + \cdots$

$$[M_X(t)]_{t=0} = e^{-\lambda}\left(1 + \frac{\lambda}{1!} + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \frac{\lambda^4}{4!} + \cdots\right)$$

$1 + \frac{\lambda}{1!} + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \frac{\lambda^4}{4!} + \cdots$ is the expansion of $\frac{\lambda^x}{x!}$ for $x = 0,1,2,\dots,n$

$$[M_X(t)]_{t=0} = e^{-\lambda}\left(\frac{\lambda^x}{x!}\right)$$

$$for\ x = 0,1,2,\dots,n$$

$$[M_X(t)]_{t=0} = \frac{\lambda^x}{x!}e^{-\lambda}\ for\ x = 0,1,2,\dots,n$$

$$[M_X(t)]_{t=0} = \sum_{x=0}^{n} \frac{\lambda^x e^{-\lambda}}{x!}$$

Therefore, it has been shown that when $t = 0$, the moment generating function for Normal distribution is the same as the Probability Mass Function for Poisson distribution.

7. CONCLUSION

Without understanding the relationship among the three distributions (Binomial, Poisson and Normal Distribution) it is difficult to understand other discrete and continuous distributions. The article has provided general definitions of the three distributions (Binomial, Poisson and Normal Distribution) and when to apply the three distributions. The article has also highlighted on the moment generating functions of the three distributions and how they are related to each other.

References

Allan. B. G, 2005. Probability Demystified. New York: The McGraw-Hill Companies, Inc.

Douglas M. C and George R. C, 2003. Applied Statistics and Probability for Engineers Third Edition. New York: John Wiley & Sons, Inc.

Hwei H.P, 1997. Theory and Problems of Probability, Random Variables, and Random Processes. The McGraw-Hill Companies, Inc.

James M.T, George B.P and Terry S, 2001. A First Course in Business Statistics Eighth Edition. Prentice Hll.

John B,2003. Engineering Mathematics Fourth Edition. Amsterdam : Elsevier Science.

John B, 2006. Higher Engineering Mathematics Fifth Edition. Amsterdam: Published by Elsevier Ltd.

Marilyn P.K and Theresa S. M 2003. Elementary Statistics from Discovery to Decision. John Wiley and Sons, INC.

Mario T. F, 2012. Elementary Statistics 11th Edition. New York: Pearson Education, Inc.

Morris D. H and Mark S. J 2012. Probability and Statistics Fourth Edition. Pearson Education, Inc.

Murry S. R and Larry S. J, 1997. Theory and Problems of Statistics Third Edition. New York: The McGraw-Hill Companies Inc.

Paul H. G, 1962. Introduction to Mathematical Statistics Third Edition. New York: John Wiley and Sons, Inc.

Post T, 2012. The Post Issue No: 246. Lusaka: The Post.

T. T. Soong. Fundamentals of Probability and Statistics for Engineers. New York: John Wiley and Sons, Ltd, (2004).